Observability Analysis of INS with a GPS Multi-Antenna System

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This paper investigates observability properties of strapdown INS aided by a GPS antenna array. The motivation to consider a GPS antenna array is that the lever-arms between the GPS antennas and IMU play an important role in the estimation of vehicle attitude and biases of IMU. It is shown that time-invariant INS error models are observable with measurements from at least three GPS antennas. It is also shown that time-varying error models are instantaneously observable with measurements from three antennas. Numerical simulation results are given to show the effectiveness of multiple GPS antennas on estimating vehicle attitude and biases of IMU when IMU has considerable magnitude of biases.

Key Words : GPS, INS, Observability analysis, IMU biases

Nomenclature

ω_{ab}^{c} : Column vector of angular velocity of frame b relative to frame a, decomposed in frame c.
P^a : Position vector decomposed in frame a.
V^a : Velocity vector decomposed in frame a.
R_a^b : Rotation matrix from frame a to frame b.
Ω_{ab}^{c} : Skew-symmetric cross product matrix of ω_{ab}^{c}.
$\hat{(\ )}$: Estimated value of ( ).
$\delta(\ )$: Estimation error of ( ).
$\dot{(\ )}$: Time derivative of ( ).
$(\ )^T$: Transpose of ( ).
$|(\ )|$: Absolute value of ( ).
$(\ )\times(\ )$: Cross product of two vectors.
I_n : $n\times n$ identity matrix.
0 : Zero matrix with an appropriate dimension.

The navigation frames used in the paper are :

i-frame : Earth-centered inertial (ECI) frame.
e-frame : Earth-centered Earth-fixed (ECEF) frame.
b-frame : Body frame (Forward, Right, Down).

1. Introduction

Improved navigation can be realized by the integration of the Global Positioning System (GPS) and inertial navigation system (INS) (Parkinson and Spilker, 1996 ; Kaplan, 1996). GPS receivers provide position and velocity of vehicles with bounded accuracy. The accuracy is independent of elapsed time from the start of measurement. However, GPS receivers can be considered as discrete-time sensors; in many cases, their sampling period is about one second. Occasionally, measurement is not available during loss of lock on satellites due to shading of GPS antennas or radio-frequency (RF) interference. The INS is a continuous- time measurement system. It is self-contained and is not dependent on the external signals. It offers short-term stability, but has poor long-term stability due to bias of an inertial measurement unit (IMU) which consists of gyros and accelerometers. Using the above complemen-

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tary properties of GPS and INS, various integration schemes have been suggested to overcome shortcomings of each sensor system (Parkinson and Spilker, 1996 ; Kaplan, 1996 ; Farrell and M. Barth, 1999).

An attractive scheme of the integration is to estimate biases of IMU in strapdown INS with GPS measurement during navigation. With this integration scheme, an accurate and low-cost navigation system that provides long-term stability and continuous measurement can be constructed with relatively low-accurate inertial sensors. There have been several approaches to the estimation of biases of IMU using GPS measurement. A technique to determine attitude of a vehicle by GPS antennas without IMU is developed in Cohen(1992), and is applied to flight test (Cohen et al., 1994). The GPS attitude measurement system in Cohen et al.(1994) is employed in Hayward et al.(1997), Gebre-Egziabher et al.(1998) to estimate gyro bias. A nonlinear observer for attitude and IMU bias is suggested in Vik et al. (1999). The observer is proven to be exponentially convergent for IMU biases that are modeled as Markov processes. A sensor fusion technique is introduced in Sadaka(1998) to estimate vehicle attitude and IMU biases with GPS, IMU, and air-data sensors.

One way to determine the effectiveness of a measurement system for estimator design is to analyze the observability properties of the system. Several methodologies have been suggested for the observability analysis of INS. The observability of INS during alignment and calibration at rest with velocity measurement is analyzed in Bar-Itzhack and Berman(1988), Jiang and Lin (1992) from a control theoretic viewpoint. Piece-wise constant modeling is proposed in Goshen-Meskin and Bar-Itzhack(1992a) for the observability analysis of time-varying systems. The piece-wise constant modeling is applied to the analysis of in-flight alignment (IFA) of INS (Goshen-Meskin and Bar-Itzhack, 1992b). Goshen-Meskin(1992b) shows that time-invariant INS error models with velocity measurement have unobservable modes which are combinations of estimation errors of attitude and IMU biases. Using the piece- wise constant modeling she shows that the number of unobservable modes can be decreased with maneuvering of the vehicle. In contrast to the works mentioned above in which observability of systems is analyzed by the rank test of observability matrix, Ham and Brown(1983) suggests error covariance matrix of the Kalman filter as a performance index for the degree of observability of systems. A nonlinear analysis based on the Lyapunov stability theorem is given for an observer design for the integration of GPS and INS (Vik et al., 1999).

In this paper, observability properties of strapdown INS aided by GPS are investigated with null space test of observability matrix. Both loosely-coupled and tightly-coupled GPS/INS integrations are considered. The INS error dynamics model is described in the ECEF frame. This model is convenient to handle GPS measurements represented in that frame. The state in the error model consists of 3-dimensional biases of gyro and accelerometer, and errors for position, velocity, and attitude. The biases of IMU are assumed to be constant. The time-constant model could be useful for biases that change very slowly compared with vehicle dynamics. Measurements for the position and velocity of GPS antennas are used for the analysis of observability properties. Multi-antenna GPS measurement systems can provide attitude information in addition to position and velocity. Since attitude can be determined from the position measurements of the antennas, attitude measurement is omitted to avoid redundancy.

Biases, scale-factor errors, and misalignment errors are usually considered as the most important errors in inertial sensors. Misalignment errors are made small during manufacturing and constant. The errors can be considered deterministic and will be neglected in the paper. Biases and scale factors of very low-grade IMU are dependent upon temperature. Since the effect of temperature on the scale factor is relatively small compared to biases (Hayward et al., 1997 ; Gebre-Egziabher et al., 1998), the scale factor is also neglected in the IMU error model. Even though biases for very low grade IMU can be described as first order Markov processes, the time constant in the Markov process model is usually very long. Hence, a time-constant model is employed for biases in IMU.

The primary motivation for considering a multi-antenna GPS measurement system is that lever arms of GPS antennas and IMU play an important role in the estimation of INS error. The measurement system cannot only make time-invariant INS error models observable, it can also make time-varying error models instantaneously observable. Hence, with the sensor system, an improvement in the on-line estimation of INS error could be expected during navigation without applying maneuvering to vehicles.

One of the main contributions of the paper is that the minimum number of GPS antennas to observe attitude and IMU biases without maneuvering of vehicle is explicitly shown to be three. The other contribution is the framework for the analysis of multi-antenna GPS aided INS. The methodology could be useful for engineers or researchers who are designing a multi-antenna GPS/INS system and require a framework for analysis of the concept.

Navigation error propagation model in the ECEF frame is given in the following section. In section 3, GPS measurement error models are described for both loosely-coupled and tightly-coupled integrations. In section 4, observability properties of GPS/INS system are given. In section 5, we present numerical simulation results which support the analysis results given in section 4. Concluding remarks are given in section 6. Finally, proofs for lemmas in section 4 are given in appendices.

2. Navigation Error Propagation Model

The navigation equations in the ECEF frame are (Wei and Schwarz, 1990)

$$\dot{P}^e = V^e \tag{1}$$

$$\dot{V}^e = R_b^e f^b - 2\omega_{ie}^e \times V^e + g^e \tag{2}$$

$$\dot{R}_b^e = R_b^e \Omega_{eb}^b \tag{3}$$

where f^b is the specific force in the body frame and g^e is the gravity in the ECEF frame. The corresponding INS mechanization differential equations are

$$\dot{\hat{P}}^e = \hat{V}^e \tag{4}$$

$$\dot{\hat{V}}^e = \hat{R}_b^e \hat{f}^b - 2\omega_{ie}^e \times \hat{V}^e + \hat{g}^e \tag{5}$$

$$\dot{\hat{R}}_b^e = \hat{R}_b^e \hat{\Omega}_{eb}^b \tag{6}$$

$$\hat{\omega}_{eb}^b = \hat{\omega}_{ib}^b - \hat{R}_e^b \omega_{ie}^e \tag{7}$$

where $\hat{f}^b$ and $\hat{\omega}_{ib}^b$ are measurements from accelerometers and gyros, respectively. The mechanization errors are modeled as

$$\hat{P}^e = P^e + \delta P \tag{8}$$

$$\hat{V}^e = V^e + \delta V \tag{9}$$

$$\hat{R}_b^e = R_b^e (I_3 + [\gamma \times]) \tag{10}$$

$$\hat{f}^b = f^b + \varepsilon_a + w_a \tag{11}$$

$$\hat{\omega}_{ib}^b = \omega_{ib}^b + \varepsilon_g + w_g \tag{12}$$

where γ is the attitude error, $[\gamma \times]$ is the cross product matrix of γ, ε_a is the accelerometer bias vector, w_a is the accelerometer noise, ε_g is the gyro bias vector, and w_g is the gyro noise. Bias vectors ε_g and ε_a are assumed to be constant. Then the linearized error propagation equations are

$$\delta \dot{P} = \delta V \tag{13}$$

$$\delta \dot{V} = G\delta P - \Omega_e \delta V - RF\gamma + R\varepsilon_a + Rw_a \tag{14}$$

$$\dot{\gamma} = -\Omega \gamma + \varepsilon_g + w_g \tag{15}$$

$$\dot{\varepsilon}_g = 0 \tag{16}$$

$$\dot{\varepsilon}_a = 0 \tag{17}$$

where $G = \frac{\partial g^e}{\partial P^e}$, F is the cross product matrix of f^b, $\Omega_e = 2\Omega_{ie}^e$, R and Ω are the simplified notations of R_b^e and Ω_{ib}^b, respectively.

3. GPS Measurement Error Model

In this section, error models of measurements for the position and velocity of each antenna are given for the analysis of observability properties of GPS/INS systems with multi-antenna. Even though multi-antenna GPS measurement systems can provide additional attitude information, the attitudes are determined from the position measurement of the antennas. Hence, attitude measurements are omitted in the measurement error model to avoid redundancy.

3.1 Loosely-coupled GPS/INS systems

In the loosely-coupled GPS/INS integrations, GPS receivers process pseudorange and pseudorange rate data to produce position and velocity. These outputs are used to correct INS errors. Measurements from the GPS receivers are modeled as

$$P_j^e = P^e + R l_j + v_{pj} \tag{18}$$

$$V_j^e = V^e + R\Omega_b l_j + v_{vj},\ j = 1, 2, \cdots, m \tag{19}$$

where P_j^e and V_j^e are position and velocity measurements from the jth GPS receiver antenna, respectively, l_j is the position of the jth GPS receiver antenna relative to that of IMU decomposed in the body frame, Ω_b is the simplified notation of Ω_{eb}^b, v_{pj} and v_{vj} are the position and velocity measurement noises of the jth GPS receiver antenna, respectively, and m is the number of GPS receiver antennas. Estimation equations for measurements are given as

$$\hat{P}_j^e = \hat{P}^e + \hat{R}_b^e l_j \tag{20}$$

$$\hat{V}_j^e = \hat{V}^e + \hat{R}_b^e \hat{\Omega}_{eb}^b l_j \tag{21}$$

The estimation errors for measurements are defined as

$$\hat{P}_j^e = P_j^e + \delta P_j^e \tag{22}$$

$$\hat{V}_j^e = V_j^e + \delta V_j^e \tag{23}$$

Then, the linearized measurement estimation errors can be shown as

$$\delta P_j^e = \delta P - R L_j \gamma - v_{pj} \tag{24}$$

$$\begin{aligned}\delta V_j^e = \delta V + R[(L_j\Omega - \Omega_b L_j)\gamma - L_j \varepsilon_g] \\ - R L_j w_g - v_{vj}\end{aligned} \tag{25}$$

where L_j is the cross product matrix of l_j.

3.1 Tightly-coupled GPS/INS systems

In the tightly-coupled integration, raw data from GPS satellites such as pseudorange and pseudorange rate are used to correct INS errors. Measurements from the GPS receivers are modeled as

$$\rho_j^{(i)} = |P^{e(i)} - P_j^e| + ct_j + \eta_j^{(i)} \tag{26}$$

$$\begin{aligned}\dot{\rho}_j^{(i)} = |V^{e(i)} - V_j^e| + c\dot{t}_j + \zeta_j^{(i)}, \\ i = 1, 2, 3, 4,\ j = 1, 2, \cdots, m\end{aligned} \tag{27}$$

where $\rho_j^{(i)}$ is the pseudorange of the ith satellite from the jth GPS receiver antenna, $P^{e(i)}$ is the position of the ith satellite decomposed in the ECEF frame, c is the speed of light, t_j is the clock bias of the jth GPS receiver, $\eta_j^{(i)}$ is the composite of errors produced by atmospheric delays, satellite ephemeris mismodeling, receiver tracking error, etc., $\dot{\rho}_j^{(i)}$ is the pseudorange rate of the ith satellite from jth GPS receiver antenna, $V^{e(i)}$ is the velocity of the ith satellite in the ECEF frame, $\dot{t}_j$ is the clock drift of the jth GPS receiver, $\zeta_j^{(i)}$ is the measurement error. Estimations for measurements are given as

$$\hat{\rho}_j^{(i)} = |P^{e(i)} - \hat{P}_j^e| + c\hat{t}_j \tag{28}$$

$$\hat{\dot{\rho}}_j^{(i)} = |V^{e(i)} - \hat{V}_j^e| + c\hat{\dot{t}}_j \tag{29}$$

The estimation errors for measurements are defined as follows:

$$\hat{\rho}_j^{(i)} = \rho_j^{(i)} + \delta\rho_j^{(i)} \tag{30}$$

$$\hat{\dot{\rho}}_j^{(i)} = \dot{\rho}_j^{(i)} + \delta\dot{\rho}_j^{(i)} \tag{31}$$

$$\hat{t}_j = t_j + \delta t_j \tag{32}$$

$$\hat{\dot{t}}_j = \dot{t}_j + \delta\dot{t}_j \tag{33}$$

Let

$$\delta\rho_j = [\delta\rho_j^{(1)}\ \delta\rho_j^{(2)}\ \delta\rho_j^{(3)}\ \delta\rho_j^{(4)}]^T \tag{34}$$

$$\delta\dot{\rho}_j = [\delta\dot{\rho}_j^{(1)}\ \delta\dot{\rho}_j^{(2)}\ \delta\dot{\rho}_j^{(3)}\ \delta\dot{\rho}_j^{(4)}]^T \tag{35}$$

$$\eta_j = [\eta_j^{(1)}\ \eta_j^{(2)}\ \eta_j^{(3)}\ \eta_j^{(4)}]^T \tag{36}$$

$$\zeta_j = [\zeta_j^{(1)}\ \zeta_j^{(2)}\ \zeta_j^{(3)}\ \zeta_j^{(4)}]^T \tag{37}$$

Then, the measurement estimation errors can be shown as

$$\delta\rho_j = H_j\begin{bmatrix}\delta P_j^e \\ c\delta t_j\end{bmatrix} - \eta_j \tag{38}$$

$$\delta\dot{\rho}_j = H_j\begin{bmatrix}\delta V_j^e \\ c\delta\dot{t}_j\end{bmatrix} - \zeta_j \tag{39}$$

where

$$H_j = \begin{bmatrix} \frac{\partial \rho_j^{(1)}}{\rho P_j^e} & 1 \\ \frac{\partial \rho_j^{(2)}}{\partial P_j^e} & 1 \\ \frac{\partial \rho_j^{(3)}}{\partial P_j^e} & 1 \\ \frac{\partial \rho_j^{(4)}}{\partial P_j^e} & 1 \end{bmatrix} \quad (40)$$

4. Observability Properties of GPS/INS Systems

In this section observability properties of GPS/INS systems are presented for both loosely-coupled and tightly-coupled integrations. Time-invariant error models as well as time-varying error models of INS are considered. The observability properties are investigated by testing null space of observability matrix. The main focus of the investigation is on the relation between the number of GPS receiver antennas and the observability of attitude error and biases of IMU.

Even though the linearized INS error models are time-varying systems, time-invariant forms of the error models are also considered in this paper for the following reason. As mentioned in Goshen-Meskin and Bar-Itzhack (1992b), time-varying INS error models can be made observable with velocity measurement by maneuvering. However, there are situations in which maneuvering of vehicles to improve observability might not be easily realizable. In these cases, the error models can be considered time-invariant. For example, an error model for vehicles that follow predetermined smooth paths can be treated as a time-invariant system; acceleration of vehicles is constant and their angular velocity is zero.

Before the main part of this section is introduced, the definition of observability of linear systems used in this paper are presented for the sake of clarity. Consider the linear system :

$$\Sigma : \dot{x}(t) = A(t)x(t)$$
$$y(t) = C(t)x(t)$$

where $A(t)$ and $C(t)$ are $n \times n$ and $p \times n$ matrices whose entries are continuous functions of t defined over $(-\infty, \infty)$

Definition 1. The dynamic equation Σ is *observable* at t_0 if there exists a finite $t_1 > t_0$ such that for any state x_0 at time t_0, the knowledge of the output $y(t)$ over the time interval $[t_0, t_1]$ suffices to determine the state x_0.

Define a sequence of $p \times n$ matrices $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ by the equation

$$N_{k+1}(t) = N_k(t)A(t) + \frac{d}{dt}N_k(t),$$
$$k = 0, 1, 2, \cdots, n-2$$
$$N_0(t) = C(t)$$

Suppose $A(t)$ and $C(t)$ in the system Σ are analytic functions of t. Then, the time-varying linear system is said to be *instantaneously observable* in $(-\infty, \infty)$ if and only if the rank of the matrix

$$\begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_{n-1}(t) \end{bmatrix}$$

is n for all t in $(-\infty, \infty)$ (Chen, 1984). If the linear time-varying system is instantaneously observable, then any state $x(t)$ can be determined from the knowledge of the measurement over an arbitrarily small interval of time for all t in $(-\infty, \infty)$. Suppose $A(t)$ and $C(t)$ in the system Σ are constant. Then, the time-invariant linear system is *observable* if and only if the rank of the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is n. If the linear time-invariant system is observable, it is observable at every initial time, and the determination of the initial state can be achieved in any nonzero time interval (Chen, 1984).

4.1 Loosely-coupled GPS/INS systems

Let

$$x = [\delta P^T \ \delta V^T \ \gamma^T \ \varepsilon_g^T \ \varepsilon_a^T]^T \quad (41)$$

$$y_n = [(\delta P_1^e)^T \ (\delta P_2^e)^T \cdots (\delta P_n^e)^T \ (\delta V_1^e)^T \ (\delta V_2^e)^T \cdots (\delta V_n^e)^T]^T \quad (42)$$

$$C_{pj}=[I_3\ 0\ -RL_j\ 0\ 0] \tag{43}$$

$$C_{vj}=[0\ I_3\ -R(\Omega_b L_j-L_j\Omega)\ -RL_j\ 0] \tag{44}$$

$$w=[0\ (Rw_a)^T\ (w_g)^T\ 0\ 0]^T \tag{45}$$

$$v_m=-[v_{p1}^T\ v_{p2}^T\ \cdots\ v_{pm}^T\ v_{v1}^T\ v_{v2}^T\ \cdots\ v_{vm}^T]^T \tag{46}$$

Then, the linearized equations of errors for INS mechanization and measurement estimation are

$$\begin{aligned}\Sigma_L:\ \dot{x}&=Ax+w\\ y_m&=C_m x+v_m\end{aligned} \tag{47}$$

where y_m is the estimation error for measurements from the m GPS receiver antennas and

$$A=\begin{bmatrix}0 & I_3 & 0 & 0 & 0\\ G & -\Omega_e & -RF & 0 & R\\ 0 & 0 & -\Omega & I_3 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{48}$$

$$C_m=[C_{p1}^T\ C_{p2}^T\ \cdots\ C_{pm}^T\ C_{v1}^T\ C_{v2}^T\ \cdots\ C_{vm}^T]^T \tag{49}$$

Let

$$T_2=\begin{bmatrix}I_3 & 0 & RL_1 & 0 & 0\\ 0 & I_3 & 0 & RL_1 & 0\\ 0 & 0 & I_3 & 0 & 0\\ 0 & 0 & \Omega & I_3 & 0\\ -R^TG & R^T\Omega_e & F-R^TGRL_1 & R^T\Omega_eRL_1-L_1\Omega & I_3\end{bmatrix} \tag{50}$$

$$\bar{x}_{\gamma1}=[0\ 0\ I_3\ 0\ 0]^T \tag{51}$$

$$\bar{x}_{\gamma2}=[0\ 0\ (l_2-l_1)^T\ 0\ 0] \tag{52}$$

$$\bar{x}_{g2}=[0\ 0\ 0\ (l_2-l_1)^T\ 0] \tag{53}$$

$$x_{\gamma1}=T_2\bar{x}_{\gamma1} \tag{54}$$

$$x_{\gamma2}=T_2\bar{x}_{\gamma2} \tag{55}$$

$$x_{g2}=T_2\bar{x}_{g2} \tag{56}$$

Let l_1, l_2, and l_3 be linearly independent, then we have the following main results:

Lemma 1. Suppose all the time-varying elements of A and C_m in the system Σ_L of (47) are analytic functions of time. Then, the time-varying system Σ_L is instantaneously observable for $m\geq 3$.

Suppose A and C_m in the system Σ_L are constant. Then we have the following lemmas:

Lemma 2. The time-invariant system Σ_L is observable for $m\geq 3$.

Lemma 3. $x_{\gamma2}$ is an unobservable mode of the time-invariant system Σ_L for $m=2$

Lemma 4. The following two conditions

(1) ω_{ib}^b is parallel with l_2-l_2,

(2) $GRl_1\times l_2=RF(l_2-l_1)$

are the only conditions for the time-invariant system Σ_L to have two unobservable modes for $m=2$. In this case, $x_{\gamma2}$ and x_{g2} are the unobservable modes.

Lemma 5. The time-invariant system Σ_L has three unobservable modes, $x_{\gamma1}$ for $m=1$.

Remark 1. The assumption that A and C_m in the system Σ_L are constant implies that $\omega_{eb}^b=0$ and $\omega_{ib}^b=\omega_{ie}^b$. This assumption also implies that G, the gradient of gravity with respect to position, is constant even though the velocity of the vehicle is not zero. This assumption can be justified if the velocity is not very fast.

Remark 2. $\bar{x}_{\gamma1}$, $\bar{x}_{\gamma2}$, and $\bar{x}_{g2}$ are unobservable modes represented in the transformed state $\bar{x}(=T_2^{-1}x)$.

The following theorem obviously follows from the above lemmas:

Theorem 1. The time-invariant system Σ_L is observable if and only if $m\geq 3$.

The proofs of the above lemmas are given in Appendix A through Appendix C.

4.2 Tightly-coupled GPS/INS systems

It is assumed that the multi-antenna measurement system is implemented by means of a single multi-antenna receiver such that

$$\begin{aligned}t_1=t_2=\cdots=t_m=t\\ \dot{t}_1=\dot{t}_2=\cdots=\dot{t}_m=\dot{t}\end{aligned}$$

Let

$$x_\rho=[x^T\ c\delta t\ c\delta\dot{t}]^T \tag{57}$$

$$y_{\rho m}=[\delta\rho_1^T\ \delta\rho_2^T\ \cdots\ \delta\rho_m^T\ \delta\dot{\rho}_1^T\ \delta\dot{\rho}_2^T\ \cdots\ \delta\dot{\rho}_m^T]^T \tag{58}$$

$$w_\rho=[w^T\ 0\ \delta\ddot{t}]^T \tag{59}$$

$$v_{\rho m}=-[\eta_1^T\ \eta_2^T\ \cdots\ \eta_m^T\ \zeta_1^T\ \zeta_2^T\ \cdots\ \zeta_m^T]^T \tag{60}$$

$$A_t=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \tag{61}$$

$$C_{ptj}=\begin{bmatrix}C_{pj} & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \tag{62}$$

$$C_{vtj}=\begin{bmatrix}C_{vj} & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{63}$$

$$C_{\rho pj}=H_jC_{ptj} \tag{64}$$

$$C_{\rho vj} = H_j C_{vtj} \tag{65}$$

where x, w, η_j, ζ_j, H_j, C_{pj}, and C_{vj} are defined in (41), (45), (36), (37), (40), (43), and (44), respectively. Then,

$$\Sigma_T : \dot{x}_\rho = A_\rho x_\rho + w_\rho \qquad y_{\rho m} = C_{\rho m} x_\rho + v_{\rho m} \tag{66}$$

with

$$A_\rho = \begin{bmatrix} A & 0 \\ 0 & A_t \end{bmatrix} \tag{67}$$

$$C_{\rho m} = [C_{\rho p1}^T \; C_{\rho p2}^T \cdots C_{\rho pm}^T \; C_{\rho v1}^T \; C_{\rho v2}^T \cdots C_{\rho vm}^T]^T \tag{68}$$

where A is given in (48). H_j in (64) and (65) consists of line of sight unit vectors from the jth antenna to the GPS satellites. According to GPS-IDC 200, orbit radii of GPS satellites are approximately 26,562 km and their angular velocities are about twice the earth's rotation rate. Hence, if the velocity of a vehicle is not very fast, H_j can be considered constant for each $j=1, 2, \cdots, m$. Let l_1, l_2, and l_3 are linearly independent. Then, we have the following lemma:

Lemma 6. Suppose H_j is a constant full rank matrix for each $j=1, 2, , m$, and all the time-varying elements of A_ρ and $C_{\rho m}$ of the system Σ_T of (66) are analytic functions of time. Then, the time-varying system Σ_T is instantaneously observable for $m \geq 3$.

Let

$$x_{\rho\gamma 1} = \begin{bmatrix} x_{\gamma 1} \\ 0 \end{bmatrix} \tag{69}$$

$$x_{\rho\gamma 2} = \begin{bmatrix} x_{\gamma 2} \\ 0 \end{bmatrix} \tag{70}$$

$$x_{\rho g2} = \begin{bmatrix} x_{g2} \\ 0 \end{bmatrix} \tag{71}$$

and l_1, l_2, and l_3 are linearly independent. Then, the following Remark 3 and Theorem 2 for the tightly-coupled integration are given without proof, since the proofs are similar to those of Lemma 2 through Lemma 5:

Remark 3. Suppose H_j is a constant full rank matrix for each $j=1, 2, \cdots, m$, and A_ρ and $C_{\rho m}$ are constant. Then, the observability conditions for the time-invariant system Σ_T of tightly-coupled integration are the same as for the time-invariant system Σ_L of (47) of loosely-coupled integration with $x_{\rho\gamma 1}$, $x_{\rho\gamma 2}$, and $x_{\rho g2}$ instead of $x_{\gamma 1}$, $x_{\gamma 2}$, and x_{g2}, respectively.

Theorem 2. Suppose H_j is a constant full rank matrix for each $j=1, 2, \cdots, m$, and A_ρ and $C_{\rho m}$ are constant. Then, the time-invariant system Σ_T is observable if and only if $m \geq 3$.

Remark 4. Let the lever arms l_1, l_2, and l_3 be linearly independent. Then, it can be shown that the time-varying INS error model is instantaneously observable with only position measurements from three GPS antennas for both loosely-coupled and tightly-coupled systems.

Remark 5. Let the lever arms l_1, l_2, and l_3 be linearly independent. Then, it can be shown that the time-invariant INS error model is observable with only position measurements from three GPS antennas for both loosely-coupled and tightly-coupled systems.

5. Simulation Results

A numerical example is given to demonstrate the behavior of Kalman filter for the estimation of attitude of a vehicle and biases of IMU with multi-antenna GPS measurement system for a very low grade IMU. Responses of the multi-rate extended Kalman filter are simulated on a simple vehicle trajectory.

The specifications of sensor errors in the numerical simulation are adopted from currently available sensors. GPS attitude determination systems are usually using four antennas with measurement frequency of 1 to 10 Hz, which is much slower than IMU measurement frequency. Very low-grade IMU error statistics are employed from the typical micro electro mechanical systems (MEMS) inertial sensors. In the simulation, estimation of navigation state is updated with the IMU measurement at 10 Hz. Since the vehicle dynamics in the simulation is very slow, fast measurement update does not improve the results and 1 Hz GPS measurement update rate and 10 Hz IMU measurement update rate are used. It is assumed that positions of GPS antennas are obtained from the double differenced carrier phase measurements. Integer ambiguity problem is assumed to be solved so that the GPS position measurements have centimeter level accuracy. Length of each lever arm of GPS antennas is about 1.7 m. All the noises in the GPS receiver and IMU are assumed to be Gaussian white. Standard deviation of GPS position measurement noise is set to [5.0 5.0 5.0] in cm. Bias and standard deviation of accelerometer noise are approximately [0.06 −0.03 0.02] and [0.01 0.01 0.01] in m/s^2. Bias and standard deviation of Gyro noise are [−0.06 0.1 −0.15] and [0.036 0.036 0.036] in degree/s. Initial estimation errors for roll, pitch, and yaw in degrees are 0.5, −0.5, and 1.0, respectively.

The path and attitude of the vehicle in the simulation are given in Fig. 1 and Fig. 2. The vehicle is motionless at start. The forward direction is north. The vehicle moves up until 1000 m, turns to the right 90 degrees, and changes attitude simultaneously during initial 250 seconds. Then, it moves to the east with constant speed afterward, without changing its altitude and attitude. It can be seen that the trajectory and attitude of the vehicle change very slowly. Estimation errors for attitude and biases are given in Fig. 3, Fig. 4, and

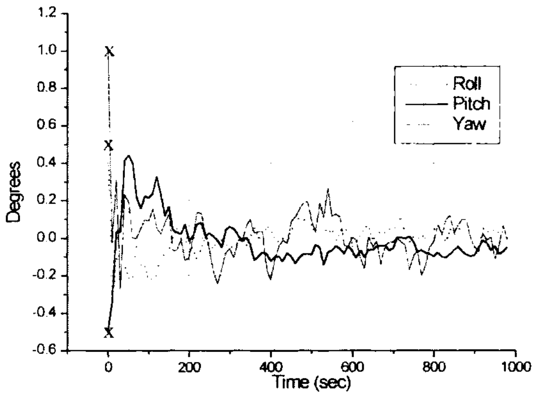

Fig. 3 Attitude estimation error

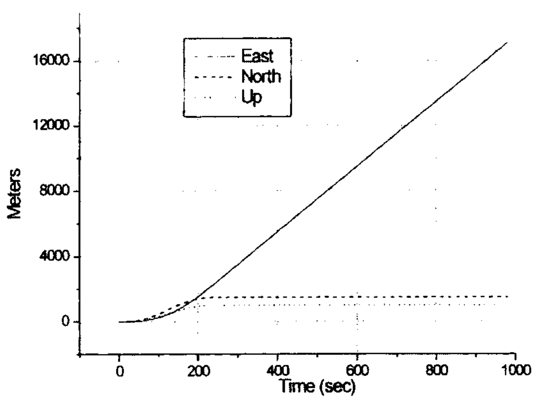

Fig. 1 Vehicle trajectory

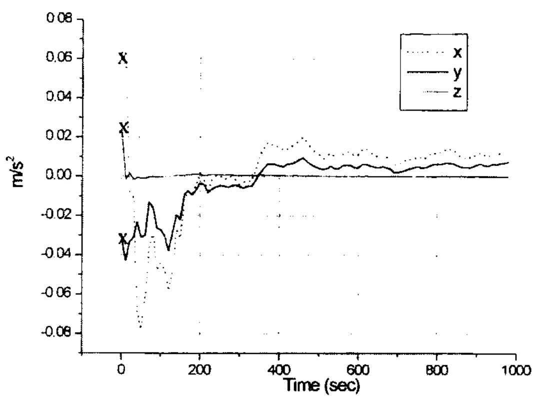

Fig. 4 Accelerometer bias extimation error

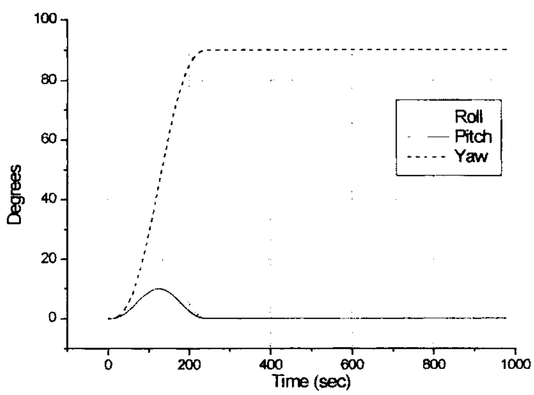

Fig. 2 Vehicle attitude

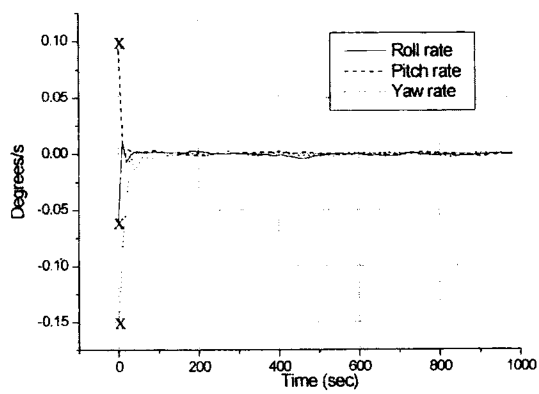

Fig. 5 Gyro bias eximation error

Fig. 5. Taking the magnitudes of the sensor noises into consideration, the GPS/INS system seems to estimate attitude of vehicle and biases of IMU quite accurately. We can see that the multi-antenna GPS measurement improves gyro bias estimation significantly.

The results show that gyro bias estimation errors converge very quickly. Since the Markov process model for very low-grade gyros usually have a time constant that is of the order of 100 seconds (Hayward et al., 1997 ; Gebre-Egziabher et al., 1998), the constant bias model in the simulation might be justified.

6. Conclusions

In this paper observability properties INS with GPS multi-antenna measurement system are presented. Estimation errors for position, velocity, attitude, and biases of inertial sensors are considered in the observability analysis. It is shown that time-invariant INS error models are observable with position measurements from at least three GPS antennas. If the number of GPS antennas is less than three, then the error models are no longer observable : There is at least one unobservable mode with position and velocity measurements from two GPS antennas. There exist at least three unobservable modes with position and velocity measurements from one GPS antenna. It is also shown that time-varying INS error models are instantaneously observable with position measurements from three antennas.

Numerical simulation results show that low-cost IMU with carrier phase differential GPS system that has *cm*-level accuracy can be an accurate and reliable navigation sensor system.

Acknowledgment

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Appendix

A.1 Proof of Lemma 1 and Lemma 2

Let $A(t)$ and $C_m(t)$ be the time-varying forms of A and C_m in (48) and (49), respectively. If noise terms are neglected, the time-varying form of the linear error model with measurement from m GPS antennas is

$$\Sigma_{Lt} : \begin{aligned} \dot{x}(t) &= A(t)x(t) \\ y(t) &= C_m(t)x(t) \end{aligned} \tag{A1}$$

with

$$A(t)=\begin{bmatrix} 0 & I_3 & 0 & 0 & 0 \\ G(t) & -\Omega_e & -R(t)F(t) & 0 & R(t) \\ 0 & 0 & -\Omega(t) & I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C_m(t)=\begin{bmatrix} C_{p1}(t) \\ \vdots \\ C_{pm}(t) \\ C_{v1}(t) \\ \vdots \\ C_{vm}(t) \end{bmatrix} \tag{A2}$$

where

$$C_{pj}(t)=[I_3\ 0\ -R(t)L_j\ 0\ 0] \tag{A3}$$

$$C_{vj}(t)=[0\ I_3\ -R(t)\{\Omega_b(t)L_j-L_j\Omega(t)\}\ -R(t)L_j\ 0],\quad j=1,2,\cdots,m \tag{A4}$$

Define

$$N_{mp}(t)=\begin{bmatrix} N_{mp0}(t) \\ N_{mp1}(t) \\ \vdots \\ N_{mp(n-1)}(t) \end{bmatrix} \tag{A5}$$

$$N_{mpv}(t)=\begin{bmatrix} N_{mpv0}(t) \\ N_{mpv1}(t) \\ \vdots \\ N_{mpv(n-1)}(t) \end{bmatrix} \tag{A6}$$

with

$$N_{mp0}(t)=\begin{bmatrix} C_{p1}(t) \\ \vdots \\ C_{pm}(t) \end{bmatrix} \tag{A7}$$

$$N_{mpv0}(t)=\begin{bmatrix} N_{mp0}(t) \\ C_{v1}(t) \\ \vdots \\ C_{vm}(t) \end{bmatrix} \tag{A8}$$

$$N_{mpj}(t)=N_{mp(j-1)}(t)A(t)+\frac{d}{dt}N_{mp(j-1)}(t) \tag{A9}$$

$$N_{mpvj}(t)=N_{mpv(j-1)}(t)A(t)+\frac{d}{dt}N_{mpv(j-1)}(t),\quad j=1,2,\cdots,n-1 \tag{A10}$$

where n is the dimension of the state of the system Σ_{Lt}. Note that

$$N_{3p0}(t)=\begin{bmatrix} I_3 & 0 & -R(t)L_1 & 0 & 0 \\ I_3 & 0 & -R(t)L_2 & 0 & 0 \\ I_3 & 0 & -R(t)L_3 & 0 & 0 \end{bmatrix} \tag{A11}$$

$$N_{3p1}(t)=\begin{bmatrix} 0 & I_3 & R(t)L_1\Omega(t)-\frac{d}{dt}(N_{3p0(1,3)}(t)) & -R(t)L_1 & 0 \\ 0 & I_3 & R(t)L_2\Omega(t)-\frac{d}{dt}(N_{3p0(2,3)}(t)) & -R(t)L_2 & 0 \\ 0 & I_3 & R(t)L_3\Omega(t)-\frac{d}{dt}(N_{3p0(3,3)}(t)) & -R(t)L_3 & 0 \end{bmatrix} \tag{A12}$$

$$N_{3p2}(t)=\begin{bmatrix} G(t) & -\Omega_e & N_{3p2(1,3)}(t) & N_{3p1(1,3)}(t)-R(t)\Omega_b(t)L_1 & R(t) \\ G(t) & -\Omega_e & N_{3p2(2,3)}(t) & N_{3p1(2,3)}(t)-R(t)\Omega_b(t)L_2 & R(t) \\ G(t) & -\Omega_e & N_{3p2(3,3)}(t) & N_{3p1(3,3)}(t)-R(t)\Omega_b(t)L_3 & R(t) \end{bmatrix} \tag{A13}$$

$$N_{3p2(i,3)}(t)=-R(t)F(t)-N_{3p1(i,3)}(t)\Omega(t)+\frac{d}{dt}(N_{3p1(i,3)}(t)) \tag{A14}$$

where $N_{3pk(i,j)}(t)$ is the (i, j) element of the matrix $N_{3pk}(t)$. Suppose $x_u(t)$ $(=[\delta P_u^T(t)\ \delta V_u^T(t)$

$\gamma_u^T(t)\ \ \varepsilon_{gu}^T(t)\ \ \varepsilon_{au}^T(t)]^T)$ is in the null space of $N_{mp}(t)$ $m\geq 3$ for any $t\in[0,\infty)$. Then,

$$N_{3pj}(t)x_u(t)=0,\ j=0,1,\cdots,n-1$$

Since $R(t)$ is nonsingular for all $t\in[0,\infty)$ and l_1, l_2, and l_3 are linearly independent, $N_{3p0}(t)x_u(t)=0$ implies that $\delta P_u(t)=\gamma_u(t)=0$. $N_{3p1}(t)x_u(t)=0$ with $\delta P_u(t)=\gamma_u(t)=0$ implies that $\delta V_u(t)=\varepsilon_g(t)=0$ for the same reason. Finally, $N_{3p2}(t)x_u(t)=0$ with $\delta P_u(t)=\delta V_u(t)=\gamma_u(t)=\varepsilon_{gu}(t)=0$ implies that $\varepsilon_{au}(t)=0$ because $R(t)$ is nonsingular. Thus $x_u(t)=0$. Hence, the rank of $N_{mp}(t)$ is n for all $t\in[0,\infty)$ and $m\geq 3$. Since the rank of $N_{mpv}(t)$ is greater than or equal to that of $N_{mp}(t)$, the rank of $N_{mpv}(t)$ is n, which is its maximum value. This completes the proof of Lemma 1. The proof of Lemma 2 is omitted because it is similar to that of Lemma 1 with the time-invariance assumption on $A(t)$ and $C_m(t)$.

A.2 Proof of Lemma 3 and Lemma 4

As mentioned in the Remark 1, the assumption that A and C_m are constant implies that $\Omega_b=0$ and $\Omega=\Omega_{ie}^b$. Consider the time-invariant system

$$\bar{\Sigma}_{L2}:\ \dot{\bar{x}}=\bar{A}\bar{x},\quad y=\bar{C}_2\bar{x} \tag{A15}$$

with

$$x=T_2\bar{x} \tag{A16}$$

$$\bar{A}=T_2^{-1}AT_2 \tag{A17}$$

$$\bar{C}_2=C_2T_2 \tag{A18}$$

where x, T_2, A, and C_2 are defined in (41), (50), (48), and (49), respectively. Note that

$$\bar{A}=\begin{bmatrix}0 & I_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & R\\ 0 & 0 & 0 & I_3 & 0\\ 0 & 0 & 0 & -\Omega & 0\\ 0 & R^TG & 0 & a_{54} & -R^T\Omega_eR\end{bmatrix} \tag{A19}$$

where

$$a_{54}=R^T(GRL_1+\Omega_eRL_1\Omega)-L_1\Omega^2-F \tag{A20}$$

Let

$$\bar{C}_2^k=\bar{C}_2\bar{A}^k,\ k=1,2,\cdots,n-1 \tag{A21}$$

where n is the size of x. Decompose $\bar{C}_2^k$ such that

$$\bar{C}_2^k=[(\bar{C}_2^k)_1\ (\bar{C}_2^k)_2\ (\bar{C}_2^k)_3\ (\bar{C}_2^k)_4\ (\bar{C}_2^k)_5] \tag{A22}$$

Then, we have the relation

$$\bar{C}_2^0=\begin{bmatrix}I_3 & 0 & 0 & 0 & 0\\ I_3 & 0 & -R(L_2-L_1) & 0 & 0\\ 0 & I_3 & 0 & 0 & 0\\ 0 & I_3 & 0 & -R(L_2-L_1) & 0\end{bmatrix} \tag{A23}$$

$$\bar{C}_2^1=\begin{bmatrix}0 & I_3 & 0 & 0 & 0\\ 0 & I_3 & 0 & -R(L_2-L_1) & 0\\ 0 & 0 & 0 & 0 & R\\ 0 & 0 & 0 & R(L_2-L_1)\Omega & R\end{bmatrix} \tag{A24}$$

$$\bar{C}_2^2=\begin{bmatrix}0 & 0 & 0 & 0 & R\\ 0 & 0 & 0 & R(L_2-L_1)\Omega & R\\ 0 & G & 0 & c_{34} & -\Omega_eR\\ 0 & G & 0 & c_{44} & -\Omega_eR\end{bmatrix} \tag{A25}$$

$$(\bar{C}_2^k)_3=0 \tag{A26}$$

$$(\bar{C}_2^k)_4=-(\bar{C}_2^{k-1})_4\Omega+(\bar{C}_2^{k-1})_5[R^T(GRL_1+\Omega_eRL_1\Omega)-L_1\Omega^2-F],\quad k=3,\cdots,n-1 \tag{A27}$$

where

$$c_{34}=GRL_1-RF-RL_1\Omega^2+\Omega_eRL_1\Omega$$
$$c_{44}=GRL_1-RF-RL_2\Omega^2+\Omega_eRL_1\Omega \tag{A28}$$

Let $\bar{x}_{2u}(=[\delta\bar{P}_{2u}^T\ \delta\bar{V}_{2u}^T\ \bar{\gamma}_{2u}^T\ \bar{\varepsilon}_{2gu}^T\ \bar{\varepsilon}_{2au}^T]^T)$ be an unobservable state of the system $\bar{\Sigma}_{L2}$. Then,

$$\bar{C}_2^k\bar{x}_{2u}=0,\ k=0,1,\cdots,n-1 \tag{A29}$$

$\bar{C}_2^0\bar{x}_{2u}=0$ implies that $\delta\bar{P}_{2u}=\delta\bar{V}_{2u}=0$, $\bar{\gamma}_{2u}=c_\gamma(l_2-l_1)$, and $\bar{\varepsilon}_{2gu}=c_g(l_2-l_1)$ where c_γ and c_g are constant numbers. $\bar{C}_2^1\bar{x}_{2u}=0$ implies that $\bar{\varepsilon}_{2au}=0$. It also implies that $\bar{\varepsilon}_{2gu}=0$ if ω_{ib}^b is not parallel with l_2-l_1. $\bar{C}_2^j\bar{x}_{2u}=0$, $j=1,2,\cdots,n-1$ with $\delta\bar{P}_{2u}=\delta\bar{V}_{2u}=\bar{\varepsilon}_{2au}=0$ implies that $\bar{\varepsilon}_{2gu}=c_g(l_2-l_1)$ if both $\omega_{ib}^b=c_w(l_2-l_1)$ and $GRl_1\times l_2=RF(l_2-l_1)$ where c_w is a constant number. Otherwise, $\bar{\varepsilon}_{2gu}=0$. Since $(\bar{C}_2^k)_3=0$, $k=1,2,\cdots,n-1$, the system $\bar{\Sigma}_{L2}$ always has one unobservable mode $\bar{x}_{\gamma 2}$. Hence, the system can have at most one additional unobservable mode $\bar{x}_{g2}$. This completes the proofs of Lemma 3 and Lemma 4.

A.3 Proof of Lemma 5

The proof of Lemma 5 is quite similar to that of Lemmas 3 and 4. Consider the time-invariant linear system

$$\bar{\Sigma}_{L1}:\ \begin{aligned}\dot{\bar{x}}&=\bar{A}\bar{x}\\ y&=\bar{C}_1\bar{x}\end{aligned} \tag{A30}$$

with

$$\bar{C}_1=C_1T_2 \tag{A31}$$

where $\bar{A}$ is defined in (A17). Define $\bar{C}_1^k$ and $(\bar{C}_1^k)_j$, $j=1, 2, \cdots, 5$ in the same way as in the Appendix A.2 Then,

$$\bar{C}_1^0=\begin{bmatrix} I_3 & 0 & 0 & 0 & 0\\ 0 & I_3 & 0 & 0 & 0\end{bmatrix} \tag{A32}$$

$$\bar{C}_1^1=\begin{bmatrix} 0 & I_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & R\end{bmatrix} \tag{A33}$$

$$\bar{C}_1^2=\begin{bmatrix} 0 & 0 & 0 & 0 & R\\ 0 & G & 0 & c_{34} & -\Omega_e R\end{bmatrix} \tag{A34}$$

$$(\bar{C}_1^k)_3=0 \tag{A35}$$

$$(\bar{C}_1^k)_4=-(\bar{C}_1^{k-1})_4\Omega+(\bar{C}_1^{k-1})_5[R^T(GRL_1+\Omega_eRL_1\Omega)-L_1\Omega^2-F],\quad k=3, \cdots, n-1 \tag{A36}$$

where c_{34} is defined in (A28). It is obvious that $[0\ 0\ I_3\ 0\ 0]^T$ is in the null space of the matrix

$$[(\bar{C}_1^0)^T\ (\bar{C}_1^1)^T\ \cdots\ (\bar{C}_1^{n-1})^T]^T \tag{A37}$$

This completes the proof of Lemma 5.

A.4 Proof of Lemma 6

Let

$$H_{\rho m}=\begin{bmatrix} H_1 & & & & & & \\ & H_2 & \ddots & & & 0 & \\ & & & H_m & & & \\ & & & & H_1 & & \\ & 0 & & & & H_2 & \ddots \\ & & & & & & H_m\end{bmatrix} \tag{A38}$$

$$C_{pvtm}(t)=[C_{pt1}^T(t)\ C_{pt2}^T(t)\ \cdots\ C_{ptm}^T(t)\ C_{vt1}^T(t)\ C_{vt2}^T(t)\ \cdots\ C_{vtm}^T(t)]^T \tag{A39}$$

where $H_{\rho m}$ is a block diagonal matrix. H_j, $j=1, 2, \cdots, m$ is defined in (40). $C_{ptj}(t)$ and $C_{vtj}(t)$, $j=1, 2, \cdots, m$, are time-varying forms of C_{ptj} and C_{vtj} in (62) and (63), respectively. Let $A_\rho(t)$ and $C_{\rho m}(t)$ be the time-varying forms of A_ρ and $C_{\rho m}$ in (67) and (68), respectively. If noise terms are neglected, the time-varying form of the system Σ_T in (66) is

$$\Sigma_{Tt}:\ \begin{aligned}\dot{x}_\rho(t)&=A_\rho(t)x_\rho(t)\\ y_{\rho m}(t)&=C_{\rho m}(t)x_\rho(t)\end{aligned} \tag{A40}$$

Then, we have

$$C_{\rho m}(t)=H_{\rho m}C_{pvtm}(t) \tag{A41}$$

Since H_j is a constant nonsingular matrix for each $j=1, 2, \cdots, H_{\rho m}$, is a constant nonsingular matrix. Hence, the procedure of instantaneous observability test based upon the null space test for the matrix pair $(A_\rho(t), C_{\rho m}(t))$ is the same as that for the matrix fair $(A_\rho(t), C_{pvtm}(t))$. The test procedure for the latter matrix pair is almost the same as in Appendix A.1 and is omitted here.